VARIABLE DIMENSION COMPLEXES, PART I:
BASIC THEORY

by

Robert M. Freund

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## Abstract

Over the past several years, much attention in the field of simplicial pivoting algorithms has been focused on the new class of variable-dimension algorithms, wherin the sequence of simplices generated varies in dimension. This study is aimed at the combinatorial nature of variable-dimension algorithms. In Part I, we introduce the notion of a V-complex, short for variable-dimension complex. We show that when a labelling function is introduced on a V-complex, certain path-following properties arise. The main result of Part I is a characterization of paths on a labelled V-complex.

Part II of this study uses the path-following theory of labelled V-complexes developed in Part I to provide constructive algorithmic proofs of a variety of combinatorial lemmas in topology. We demonstrate two new dual lemmas on the n-dimensional cube, and use a Generalized Sperner Lemma to prove a generalization of the Knaster-Kuratowski-Mazurkiewicz Covering Lemma on the simplex. We also show that Tucker's Lemma can be derived directly from the Borsuk-Ulam Theorem. We report the interrelationships between these results, Brouwer's Fixed Point Theorem, and the existence of stationary points on the simplex.

Key Words

complementary pivot, fixed points, simplex, pseudomanifold, orientation, stationary point, combinatorial topology, V-complex, antipodal point.

## Introduction

Over the past several years, much attention in the field of simplicial pivoting algorithms has been focused on the new class of variable-dimension algorithms, wherein the sequence of simplices generated varies in dimension, see van der Laan and Talman [12, 13, 14, 15], Reiser [20], Luthi [19], and Kojima and Yamamoto [9]. Like other simplicial algorithms, these algorithms usually can be executed with integer or vector labels.

When vector labels are used, it is appropriate to interpret these algorithms as piecewise-linear (PWL) path tracing as in Eaves [2]. Kojima and Yamamoto in [9] present a theory of dual pairs of subdivided manifolds that is a basis for interpreting these algorithms via PWL path tracing.

Although the integer-labelled variable dimension algorithms can also be interpreted as PWL path tracing, it seems more appropriate to use a combinatorial framework in which to study these algorithms. The integer-labelled algorithms usually are associated with some combinatorial lemma in topology, e.g. Sperner's Lemma, and the algorithms result in constructive proofs.

This study is aimed at the combinatorial nature of variable-dimension algorithms. In Part I, we introduce the notion of a V-complex, short for variable-dimension complex. We show that when a labelling function is introduced on a V-complex, certain path-following properties arise. The main result of Part I is a characterization of paths on a labelled V-complex. Whereas most of the variable-dimension integer-labelled algorithms can be interpreted as path-following on a V-complex, see Freund [6], our interest here lies in the study of some combinatorial lemmas in topology.

In Part II of this study, we give constructive, algorithmic proofs to a number of combinatorial lemmas in topology, namely Sperner's Lemma [26], Kuhn's Cubical Lemma [10], Scarf's Dual Sperner Lemma [22], a Generalized Sperner Lemma [4], Tucker's Lemma on the cube [29], Gale's Hex Theorem [7], and two new dual lemmas on cube [6]. We also show the variety of relationships between these lemmas, the Brouwer fixed point theorem, and the Borsuk-Ulam Antipodal Point Theorem, and a new set covering theorem.

This study is based on the author's doctoral disseration [6].

## Notation

Let $\mathbb{R}^n$ be real n-dimensional space, and let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$. Define e to be the vector of 1's, namely $e = (1, \ldots, 1)$. The empty set is represented by $\phi$. For two sets S and T, define the symmetic difference operator $S \Delta T = \{x \mid x \in S \cup T,\ x \notin S \cap T\}$ and define $S \setminus T = \{x \mid x \in S,\ x \notin T\}$.
For $x \in \mathbb{R}^n$, $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$, $\|x\|_\infty = \max_i |x_i|$, the Euclidean and maximum norms, respectively.
Let $e^i$ be the $i^{th}$ unit vector in $\mathbb{R}^n$. For a matrix A, let $A_{.i}$ be the $i^{th}$ column of A, and $A_{i.}$ be the $i^{th}$ row of A.

## Preliminaries - Complexes, Pseudomanifolds, Orientation, Triangulations

An abstract complex consists of a set of vertices $K^0$ and a set of finite nonempty subsets of $K^0$, denoted K, such that

i) $v \in K^0$ implies $\{v\} \in K$

ii) $\emptyset \neq x \subset y \in K$ implies $x \in K$.

The elements of K are called simplices. Suppose $x \in K$ and $|x| = n+1$, where $|.|$ denotes cardinality. Then x is called an n-dimensional simplex, or simply an n-simplex. Condition (i) above means that all members of $K^0$ are 0-simplices, and condition (ii) means that K is closed under subsets. Technically, an abstract complex is defined by the pair $(K,K^0)$. However, since the set $K^0$ is implied by K, it is convenient to simply denote the complex by K alone.

An abstract complex K is said to be finite if the set $K^0$ is finite. An abstract complex K is said to be locally finite if for each $v \in K^0$, the set of simplices containing v is a finite set. More formally, K is locally finite if and only if for each $v \in K^0$,

$$\{x \in K | v \in x\} \text{ is a finite set.}$$

A subset L of K is said to be a subcomplex of K if L itself is a complex.

A particular class of complexes, called pseudomanifolds, is central to the theory to be developed. An n-dimensional pseudomanifold (where $n \geq 1$) or more simply an n-pseudomanifold, is a complex K such that

i) $x \in K$ implies there exists $y \in K$ with $|y| = n+1$, and $x \subset y$.

ii) If $x \in K$ and $|x| = n$, then there are at most two n-simplices that contain x.

Let K be an n-pseudomanifold. The boundary of K, denoted $\partial K$, is defined to be the set of simplicies $x \in K$ such that x is contained in an (n-1)-simplex $y \in K$, and y is a subset of exactly one n-simplex of K.

An n-pseudomanifold K is said to be homogeneous if for any pair of n-simplices $x, y \in K$, there is a finite sequence $x = x_1, x_2, x_3, \ldots, x_m = y$ of n-simplices in K such that $x_i \cap x_{i+1}$ is an (n-1)-simplex in K, for $i = 1, \ldots, m-1$.

Let K be an n-pseudomanifold and let x be an n-simplex in K. Let $x = \{v_0, \ldots, v_n\}$. Let $y = \{v_1, \ldots, v_n\}$. If $y \notin \partial K$, there is unique $w \in K^0$ such that $\{w, v_1, \ldots, v_n\}$ is an n-simplex in K. The process of exchanging $v_0$ for w to obtain a new n-simplex is called a pivot. In general, if x and z are n-simplices and z can be obtained from x by a pivot, x and z are said to be a neighboring pair, or simply neighbors.

For the purposes of this study a 0-dimensional pseudomanifold K is defined to be a set of one of the following two forms:

i) $K = \{\emptyset, \{a\}\}$, where $K^0 = \{a\}$, or

ii) $K = \{\emptyset, \{a\}, \{b\}\}$, where $K^0 = \{a,b\}$.

Note that K contains $\emptyset$, the empty set, as a member, and so is not properly a complex, by the usual definition. Here, however, $\emptyset$ is a -1-simplex. If K is of type (i), then $\partial K = \{\emptyset\}$. If K is of type (ii), then $\partial K = \emptyset$, i.e. K has no boundary, and {a} and {b} are neighbors.

Let K be a homogeneous n-pseudomanifold, and let x be an n-simplex in K. Let $(v_0, \ldots, v_n)$ be some fixed ordering of the vertices of x. For an arbitrary ordering $(v_{j_0}, \ldots, v_{j_n})$ of x, this ordering is said to have a (+) orientation if and only if the permutation

$$(j_0, \ldots, j_n)$$

is even; otherwise the orientation is (-).

Now let us extend this notion to all of K. Fix an ordering of all n-simplices of K. Let x be an n-simplex and let y be an n-simplex obtained by pivoting on an element $v_{j_i}$ of x and replacing $v_{j_i}$ by w. We say that the pair (x,y) is <u>coherently-oriented</u> if the orderings $(v_{j_0}, \ldots, v_{j_n})$ and $(v_{j_0}, \ldots, v_{j_{i-1}}, w, v_{j_{i+1}}, \ldots, v_{j_n})$ are differently oriented, i.e. one is (+) and the other is (-). K is said to be <u>orientable</u> if it is possible to specify orientations on all n-simplices of K in a way that all neighboring n-simplices x, y are coherently-oriented.

Finally, we define induced orientation on the boundary of K. Let K be a homogeneous orientable n-pseudomanifold such that $\partial K$ is not empty. Let y be an (n-1)-simplex in $\partial K$. Then there is a unique n-simplex $x \in K$ such that $y \subset x$. Orient K coherently. Let $(v_{j_0}, \ldots, v_{j_n})$ be an ordering of the vertices of x. $y = x \setminus \{v_{j_i}\}$ for some i uniquely determined. Denote the orientation of the ordering $(v_{j_0}, \ldots, v_{j_n})$ by $Or(v_{j_0}, \ldots, v_{j_n})$. Then we define the <u>induced orientation</u> on y as

$$Or(v_{j_0}, \ldots, v_{j_{i-1}}, v_{j_{i+1}}, \ldots, v_{j_n}) = (-1)^i \, Or(v_{j_0}, \ldots, v_{j_n}).$$

Proposition 1  Induced orientation is well-defined.

PROOF. Let y be an (n-1)-simplex in $\partial K$, and let x be the unique n-simplex in K that contains y. Let $(i_0, \ldots, i_{n-1})$ be an ordering of the vertices of y, and let $(j_0, \ldots, j_n)$ and $(\ell_0, \ldots, \ell_n)$ be orderings of the vertices of x, from which $(i_0, \ldots, i_{n-1})$ is derived. $y = x \setminus \bar{v}$ for some unique $\bar{v} \in x$. $\bar{v} = v_{j_r} = v_{\ell_s}$ for some unique r, s.
If r = s, then $(j_0, \ldots, j_n) = (\ell_0, \ldots, \ell_n)$, and

$$\mathrm{Or}(i_0, \ldots, i_{n-1}) = (-1)^r\, \mathrm{Or}(j_0, \ldots, j_n) = (-1)^s\, \mathrm{Or}(\ell_0, \ldots, \ell_n)$$

trivially.

So suppose s > r. It takes s-r transpositions to change $(j_0, \ldots, j_n)$ to $(\ell_0, \ldots, \ell_n)$. Hence $(-1)^r\, \mathrm{Or}(j_0, \ldots, j_n)$ $= (-1)^r\, (-1)^{s-r}\, \mathrm{Or}(\ell_0, \ldots, \ell_n) = (-1)^s\, \mathrm{Or}(\ell_0, \ldots, \ell_n)$. ⊗

An n-dimensional pseudomanifold is an abstraction of a triangulation of an n-dimensional set in $\mathbb{R}^n$. The m-simplices of pseudomanifolds correspond to geometric objects, which are also called m-simplices. Let $v^0, \ldots, v^m$ be vectors in $\mathbb{R}^n$. $v^0, \ldots, v^m$ are said to be affinely independent if the matrix

$$\begin{bmatrix} v^0 & \ldots & v^m \\ 1 & \ldots & 1 \end{bmatrix}$$

has rank m+1. If $v^0, \ldots, v^m$ are affinely independent then their convex hull, denoted $\langle v^0, \ldots v^m \rangle$ is said to be an m-dimensional simplex, or more simply an m-simplex. All m-simplices are closed and bounded polyhedral convex sets. Let $\{v^{j_0}, \ldots, v^{j_k}\}$ be a subset of $\{v^0, \ldots, v^m\}$. Then $\langle v^{j_0}, \ldots, v^{j_k} \rangle$ is called a k-dimensional face of k-face of $\langle v^0, \ldots, v^n \rangle$. Any k-face of $\langle v^0, \ldots, v^m \rangle$ is a k-simplex itself. An (m-1)-face of an m-simplex is called a facet of the m-simplex.

Let H be an m-dimensional convex set in $IR^n$. Let C be a collection of m-simplices $\sigma$ together with all of their faces. C is a triangulation of H if

i) $H = \bigcup_{\sigma \in C} \sigma$

ii) $\sigma, \tau \in C$ imply $\sigma \cap \tau \in C$

iii) If $\sigma$ is an (m-1)-simplex of C, $\sigma$ is a face of at most two m-simplices of C.

The connection between triangulations and pseudomanifolds should be clear. Corresponding to each simplex $\sigma$ in C is its set of vertices $\{v^0, \ldots, v^k\}$. Let K be the collection of these sets of vertices together with their nonempty subsets. Then K is an m-dimensional pseudomanifold.

C is said to be locally finite if for each vertex $v \in H$, the set of simplicies $\sigma \in C$ that contain v is a finite set.

Pertinent references for complexes and pseudomanifolds are Lefschetz [16] or Spanier [25]. Some of the material on orientation was taken from Lemke and Grotzinger [18]. The notion of orienting pseudo-manifolds can be extended to triangulations by the use of determinants. The interested reader can refer to Eaves [2] and Eaves and Scarf [3] for an exposition on orienting triangulations.

V-Complexes

This section defines a particular combinatorial framework called a V-complex that is used in the study of combinatorial lemmas in topology, as well as in the study of variable-dimension simplicial pivoting algorithms for obtaining solutions of equations. When applied to a triangulation of a set S in $IR^n$, a V-complex constitutes a division of S into a number of regions of varying dimension. As an example, consider the set $S = \{x \in IR^2 | -e \leq x \leq e\}$ triangulated in such a manner that each coordinate axis is also triangulated. Consider the regional division defined as follows:

$R(\emptyset) = \{0\}$
$R(\{1\}) = \{ x \in S \mid x_1 \geq 0,\ x_2 = 0 \}$
$R(\{-1\}) = \{ x \in S \mid x_1 \leq 0,\ x_2 = 0 \}$
$R(\{2\}) = \{ x \in S \mid x_1 = 0,\ x_2 \geq 0 \}$
$R(\{-2\}) = \{ x \in S \mid x_1 = 0,\ x_2 \leq 0 \}$
$R(\{1,2\}) = \{ x \in S \mid x_1 \geq 0,\ x_2 \geq 0\}$
$R(\{1,-2\}) = \{ x \in S \mid x_1 \geq 0,\ x_2 \leq 0\}$
$R(\{-1,2\}) = \{ x \in S \mid x_1 \leq 0,\ x_2 \geq 0\}$
$R(\{-1,2\}) = \{ x \in S \mid x_1 \leq 0,\ x_2 \leq 0\}$

Let us define $\mathcal{T}$ to be the domain over which R(.) is defined, namely $\mathcal{T} = \{\emptyset, \{1\}, \{-1\}, \{2\}, \{-2\}, \{1,2\}, \{1,-2\}, \{-1,2\}, \{-1,-2,\}\}$, and note that for $S, T \in \mathcal{T}$, $S \cap T \in \mathcal{T}$, and $R(S \cap T) = R(S) \cap R(T)$. Also note that each R(T) is a $|T|$-dimensional manifold. For a given $T \in \mathcal{T}$, suppose there is a $j \notin T$ such that $T \cup \{j\} \in \mathcal{T}$. Then, because C triangulates each region R(.), each $|T|$-simplex $x = \langle v^\circ, \ldots, v^{|T|} \rangle$ in R(T) will have a unique vertex $\bar{v}$ in $R(T \cup \{j\})$ such that $\langle v^\circ, \ldots, v^{|T|}, \bar{v} \rangle$ is a $(|T| + 1)$ - simplex in $R(T \cup \{j\})$.

Because the above-mentioned properties do not depend on the particular triangulation of S and are more combinatoric than geometric, it is convenient to restate them in the more abstract framework of complexes and pseudomanifolds. Let K be the pseudomanifold corresponding to C, and $K^\circ$ its set of vertices. For each $T \in \mathcal{T}$, define A(T) to be the pseudomanifold corresponding to the restriction of C to R(T). The above properties of R(.) carry over to A(.), namely $A(S \cap T) = A(S) \cap A(T)$ for $S, T \in \mathcal{T}$, and A(T) is a $|T|$- dimensional pseudomanifold. Furthermore, if $x = \{v^\circ, \ldots, v^{|T|}\}$ is a $|T|$-dimensional simplex in A(T) and $j \notin T$ but $T \cup \{j\} \in \mathcal{T}$, then there is a unique vertex $\bar{v}$ of $K^\circ$ such that $\{ v^\circ, \ldots, v^{|T|}, \bar{v}\} \in A(T \cup \{j\})$.

In a typical application of a V-complex, we have a function L(.) that assigns a label to each vertex v of the triangulation. In the above example consider a function L(.): $K^{\circ} \to N$, where $N=\{1,2,-1,-2\}$. The properties of A(.), $\mathcal{J}$, and N, which are formalized in a more general setting below, will be central to the development of algorithms which will find a simplex each of whose labels have certain desirable properties. We now proceed to define a V-complex.

Let K be a locally finite simplicial complex with vertices $K^{0}$. Let N be a fixed finite nonempty set, which we call the label set. Let $\mathcal{J}$ denote a collection of subsets of N, which we call the admissible subsets of N. Let A(.) be a set-to-set map, $A: \mathcal{J} \to 2^{K} \setminus \{\emptyset\}$, where $2^{S}$ denotes the collection of subsets of a set S. K is said to be a V-complex with operater A(·) and admissable sets $\mathcal{J}$, if the following eight conditions are met:

i) K is a locally finite complex with vertices $K^0$

ii) $\mathcal{T} \subset 2^N$

iii) $T \in \mathcal{T}$, $S \in \mathcal{T}$ implies $S \cap T \in \mathcal{T}$

iv) $A(\cdot) : \mathcal{T} \to 2^K \setminus \{\emptyset\}$

v) For any $x \in K$, there is a $T \in \mathcal{T}$ such that $x \in A(T)$

vi) For any $S, T \in \mathcal{T}$, $A(S \cap T) = A(S) \cap A(T)$

vii) For $T \in \mathcal{T}$, $A(T)$ is a subcomplex of K and is a pseudomanifold of dimension $|T|$, where $|\cdot|$ denotes the cardinality of the set.

viii) $T \in \mathcal{T}$, $T \cup \{j\} \in \mathcal{T}$, $j \notin T$ implies $A(T) \subset \partial A(T \cup \{j\})$.

Let us examine these properties. (i), (ii) and (iv) reiterate what has been said in the preceding paragraph. (iii) imposes some structure on $\mathcal{T}$, namely that it is closed under intersections. (v) states that the map $A(\cdot)$ covers all simplices of K. (vi) states that $A(\cdot)$ is a homomorphism with respect to intersections. (vii) states that each $A(T)$ is an appropriately-dimensioned pseudomanifold. Condition (viii) stipulates how the pseudomanifolds $A(T)$ are arranged relative to each other, namely that $A(T)$ is part of the boundary of $A(T \cup \{j\})$.

The nomenclature "V-complex" is short for variable-dimension complex, and derives from property (vii) above, where the dimension of the pseudomanifolds $A(T)$ varies over the range of T in $\mathcal{T}$.

As an example of a V-complex, consider a triangulation C of $\mathbb{R}^2$ that also triangulates the coordinate axes. Let $N = \{\pm 1, \pm 2\}$. Let $\mathcal{T}$ be the collection of sets $\{1\}$, $\{-1\}$, $\{2\}$, $\{-2\}$, $\{1,2\}$, $\{1,-2\}$, $\{-1,2\}$, $\{-1,-2\}$ and $\emptyset$. Let K be the complex (actually a pseudomanifold itself) corresponding to the triangulation C. Then for each $T \in \mathcal{T}$, we define

$$A(T) = \{x \in K \mid v \in x \text{ implies } i \cdot v_{|i|} \geq 0 \text{ for each } i \in T, \text{ and } v_i = 0 \text{ if } i \notin T \text{ and } -i \notin T\}.$$

Figure 1(a) illustrates this V-complex. Note that for matters of convenience, the set brackets { } have been deleted. In the figure, $A(\emptyset)$ is the origin, $A(i)$ corresponds to one of the four axes emanating from the origin, and $A(i,j)$ corresponds to one of the four quadrants.

Figure 1(a) is by no means the only V-complex associated with $\mathbb{R}^2$. Figures 1(b) and 1(c) demonstrate other V-complexes associated with $\mathbb{R}^2$, with the triangulations omitted.

Suppose $K$ is a V-complex. Let $x \in K$. We define

$$T_x = \bigcap_{\substack{T \in \mathcal{T} \\ x \in A(T)}} T$$

$T_x$ then is the smallest set $T$ such that $x \in A(T)$. We say $x$ is <u>full</u> if $|x| = |T_x| + 1$. $x$ is a <u>full simplex</u> if it is a maximum-dimension simplex in $A(T_x)$.

For each $T \in \mathcal{T}$, we also define $\partial' A(T)$ as

$$\partial' A(T) = \{x \in \partial A(T) \,|\, T_x = T\}.$$

We illustrate the above definitions in the V-complex in Figure 2. In the figure, the left-most vertex of the 2-simplex is $A(\emptyset)$, the "bottom" line segment is $A(1)$, the left-sided line segment is $A(2)$, and the simplex itself is $A(1,2)$.

For $x = \{d,e\}$, $T_x = \{1,2\}$, for $x = \{f,g\}$, $T_x = \{1\}$. For $x = \{e,f,h\}$, $T_x = \{1,2\}$. The simplices $\{a\}$, $\{f,g\}$, and $\{e,h,f\}$ are all full. We have $\partial' A(1) = \{c\}$, $\partial' A(2) = \{b\}$, and $\partial' A(1,2)$ is the pseudomanifold corresponding to the line segment from b to c. Thus, while both $\{k,\ell\}$ and $\{f,g\}$ are elements of $\partial A(1,2)$, $\{k,\ell\} \in \partial' A(1,2)$, whereas $\{f,g\} \notin \partial' A(1,2)$.

For $T = \emptyset$, $A(T)$ contains only one vertex, the origin, and the empty set $\emptyset$, and therefore $\partial' A(\emptyset) = \{\emptyset\}$.

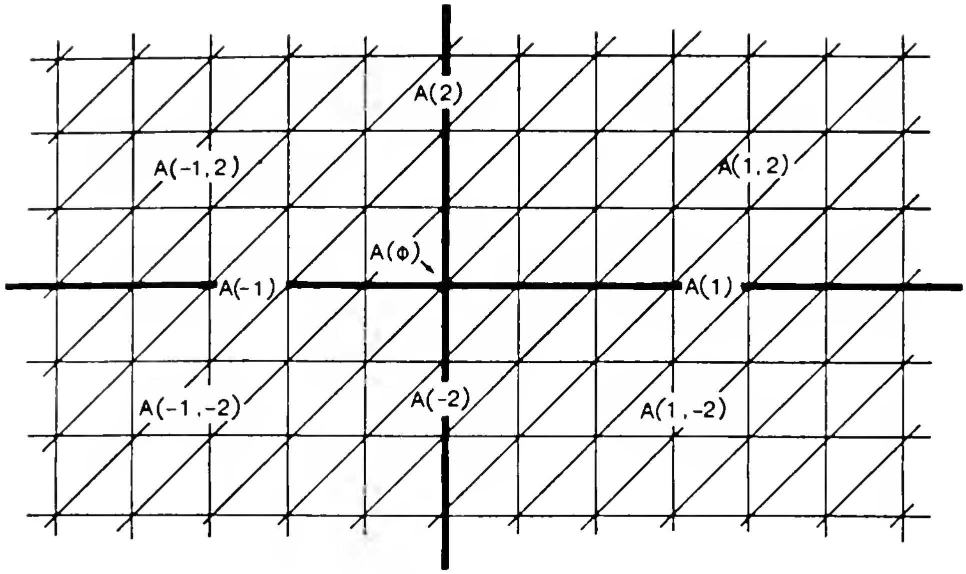

Figure 1(a)

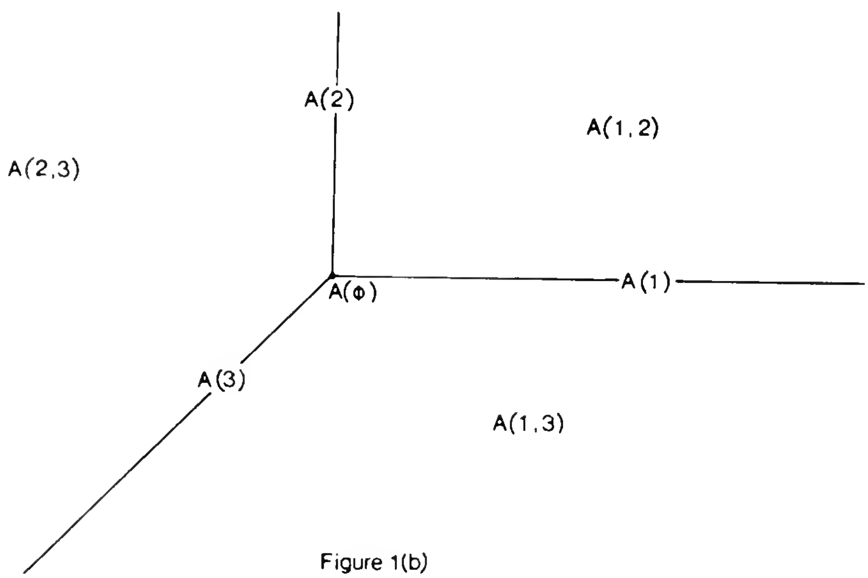

Figure 1(b)

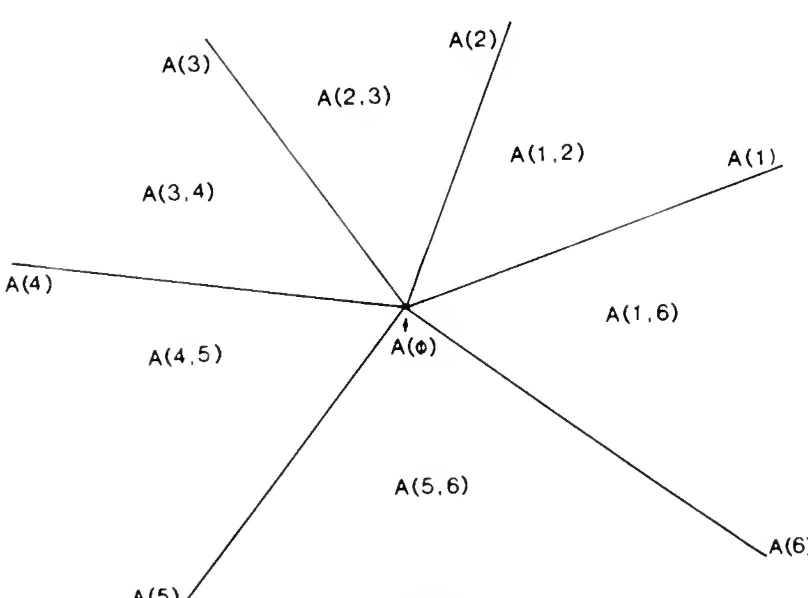

Figure 1(c)

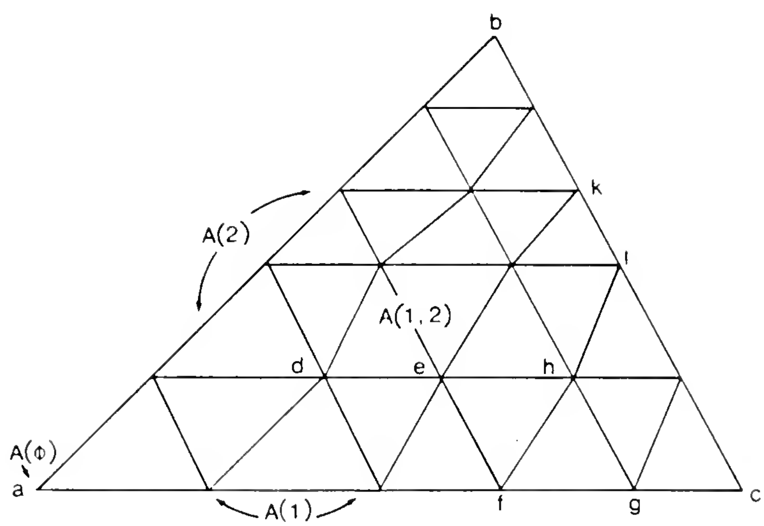


Figure 2

Let $K$ be a V-complex with label set $N$. Let $L(.)$ be a function that assigns to each $v \in K^0$ an element $i \in N$. Such a function $L(.)$ is a <u>labelling function</u>. For a simplex $x = \{v^0, \ldots, v^m\} \in K$, we define $L(x) = \bigcup_{v \in x} L(v)$. $L(x)$ is the set of labels spanned by the elements of $x$.

We define two distinct simplices $x,y \in K$ to be <u>adjacent</u> (written $x \sim y$) if

i) $x$ and $y$ are full

and ii) $L(x \cap y) = T_x \cup T_y$.

Note that adjacency is symmetric: $x \sim y$ if and only if $y \sim x$. Also note that if $x \sim y$ for some $y$, $L(x) \supset T_x$. To see this, observe that if $x \sim y$, $L(x) \supset L(x \cap y) = T_x \cup T_y \supset T_x$.

Figure 3 represents a V-complex whose vertices $K^0$ have labels $L(.)$. In the figure, we have the following adjacent simplices:

$\{a\} \sim \{a,b\} \sim \{b,c\} \sim \{c,d\} \sim \{c,d,u\} \sim \{u,d,w\} \sim \{u,w,v\}$

$\{u,s,v\} \sim \{s,u,t\} \sim \{t,s,p\} \sim \{p,t,q\} \sim \{p,q\} \sim \{p,n\} \sim \{n,m\} \sim \{n,m,r\} \sim \{m,r,k\} \sim \{k,r,s\} \sim \{s,k,v\}$

$\{h,g,w\} \sim \{g,w,e\} \sim \{w,e,d\} \sim \{d,e\} \sim \{e,f\}$.

Observe that, in the figure, any full simplex is adjacent to at most two other simplices. We shall see later on that this is true in general. Observe also that the adjacency relationship results in the formation of three distinct "paths" of simplices, each path being a string of simplices adjacent to one another.

The purpose of the remainder of Part I of this study is to give a characterization of these paths. However, we must develop the theory of V-complexes further before a complete characterization is possible.

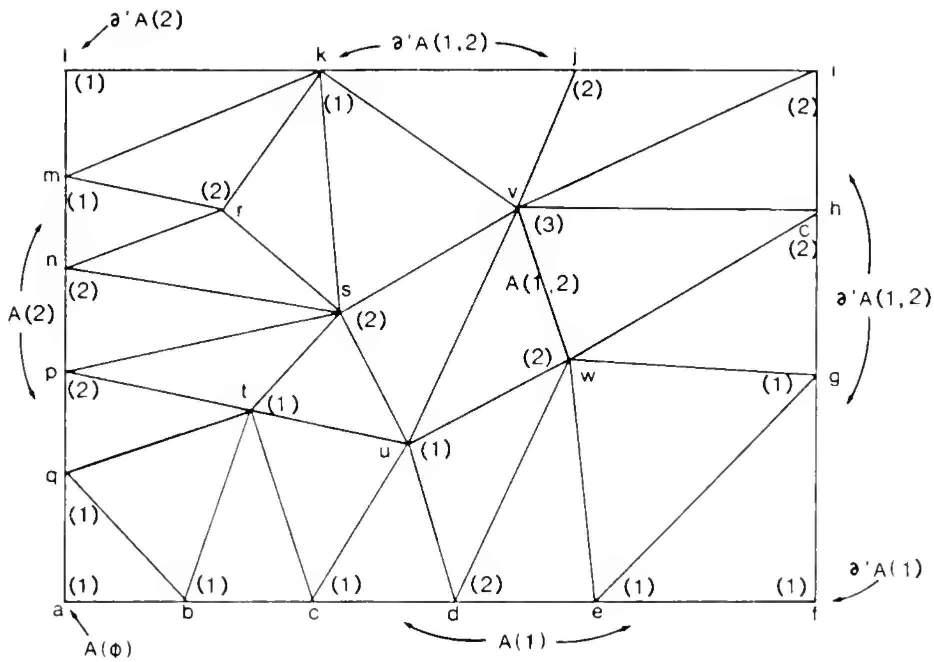


Figure 3

H-Complexes

Let K be a V-complex with label set $N$ and admissible sets $\mathcal{J}$. We wish to "lift" $K$ into a pseudomanifold of dimension $n$ where $n = |N|$. Without loss of generality, assume $N = \{1, \ldots, n\}$. Let $K^0$ be the set of vertices of $K$. We define artifical vertices $q_1, \ldots, q_n$. Let $\bar{Q} = \{q_1, \ldots, q_n\}$. Define $Q_T = \{q_i \in \bar{Q} | i \in N \setminus T\}$. The H-complex $\bar{K}$ associated with the V-complex K is defined

$$\bar{K} = \{x \cup Q | x \cup Q \neq \emptyset, x \in K, Q \subset Q_{T_x}\},$$

where $\bar{K}^\circ$ is the set of 0-simplices of $\bar{K}$. The nomenclature "H-complex" is short for homogeneous-dimension complex and derives from Theorem 2, below, which states that $\bar{K}$ is an n-dimensional pseudomanifold, i.e., its dimension does not vary.

Theorem 2. $\bar{K}$ is an n-dimensional pseudomanifold.

PROOF. Clearly $\bar{K}$ is closed under nonempty subsets, and so is a complex. Let $x \cup Q \in \bar{K}$. Then there exists $y \in A(T_x)$ that is full and $y \supset x$. Let $P = Q_{T_x}$. Then we have $x \cup Q \subset y \cup P \in \bar{K}$. Furthermore,

$$|y \cup P| = |y| + |P| = |y| + n - |T_x| = |y| + n - (|y|-1) = n+1.$$

Therefore every simplex in $\bar{K}$ is a subset of an n-simplex in $\bar{K}$. It only remains to show that each (n-1)-simplex of $\bar{K}$ is contained in at most two n-simplices.

Let $\bar{x} = x \cup Q_x$ be an n-simplex in $\bar{K}$, and let $\tilde{y} \subset \bar{x}$ be an (n-1)-simplex in $\bar{K}$. Suppose $\tilde{y} \subset \bar{z} \neq \bar{x}$, and $\bar{z}$ is an n-simplex in $\bar{K}$. We aim to show that $\bar{z}$ is uniquely determined by $\bar{x}$ and $\tilde{y}$. Since $\bar{x}$ is an n-simplex, $x$ is full and $Q_x = Q_{T_x}$. We have three cases:

Case 1. $\tilde{y} = \bar{x} \setminus \{q_i\}$ for some $q_i \in Q_x$. Let $\bar{z} = z \cup Q_z$. If $z = x$, then $Q_z = Q_x$, and so $\bar{z} = \bar{x}$, a contradiction. Therefore $z \neq x$. But since $z \supset x$, we must have $z = x \cup \{w\}$ for some w. Therefore $Q_z = Q_x \setminus \{q_i\}$, and so $T_z = T_x \cup \{i\}$. By property (viii) of V-complexes, the choice of w, and hence $\bar{z}$, is unique.

Case 2. $\tilde{y} = \bar{x} \setminus \{v\}$ for some $v \in x$, and $x \setminus \{v\}$ is not full. We can write $\tilde{y} = y \cup Q_x$ where $y = x \setminus \{v\}$. Since y is not full, we must have $z = y \cup \{w\}$ for some $w \in K^0$, $w \notin y$, and hence $Q_z = Q_x$, whence $T_z = T_x$. The choice of w is uniquely determined, since $A(T_x)$ is a pseudomanifold.

Case 3. $\tilde{y} = \bar{x} \setminus \{v\}$ for some $v \in x$, and $x \setminus \{v\}$ is full. Again we write $\tilde{y} = y \cup Q_x$, where $y = x \setminus \{v\}$. Since y is full, $T_y = T_x \setminus \{i\}$ for some $i \in T_x$, and by property (viii) of V-complexes, $y \in \partial A(T_x)$. Hence we cannot have $z = y \cup \{w\}$ for any $w \in K^0$. Therefore the unique n-simplex of $\bar{K}$ containing $\tilde{y}$ is $\bar{z} = y \cup Q_x \cup \{q_i\}$. ⊗

We illustrate this result in Figure 4.
In Figure 4(c), $A(\emptyset)$ consists of the north and south "poles" of the circle and $A(1)$, $A(2)$ are the right and left arcs, respectively.

Our next task is a characterization of the boundary of $\bar{K}$.

Lemma 3. $\partial\bar{K} = S_1 \cup S_2$, where

$$S_1 = \{y \cup Q_y \in \bar{K} \mid y \in \partial' A(T_y)\}, \text{ and}$$

$$S_2 = \{y \cup Q_y \in \bar{K} \mid N \setminus \{i \mid q_i \in Q_y\} \notin \mathcal{J}\}.$$

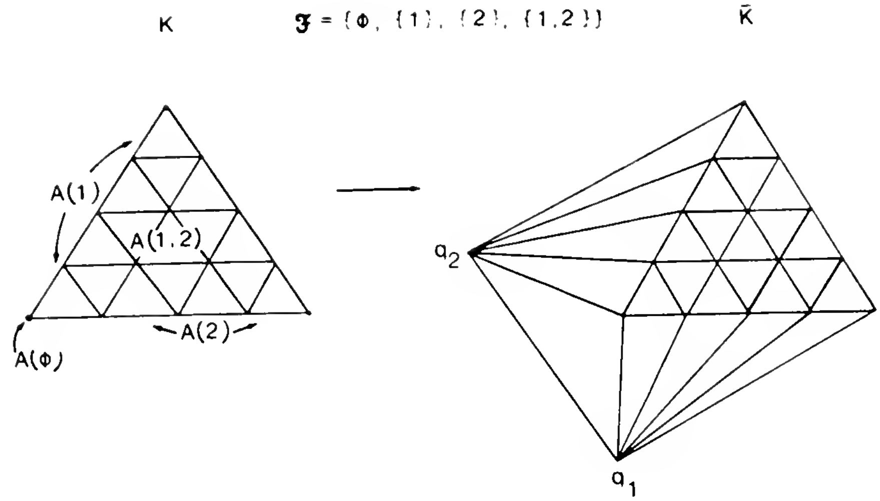


Figure 4(a)

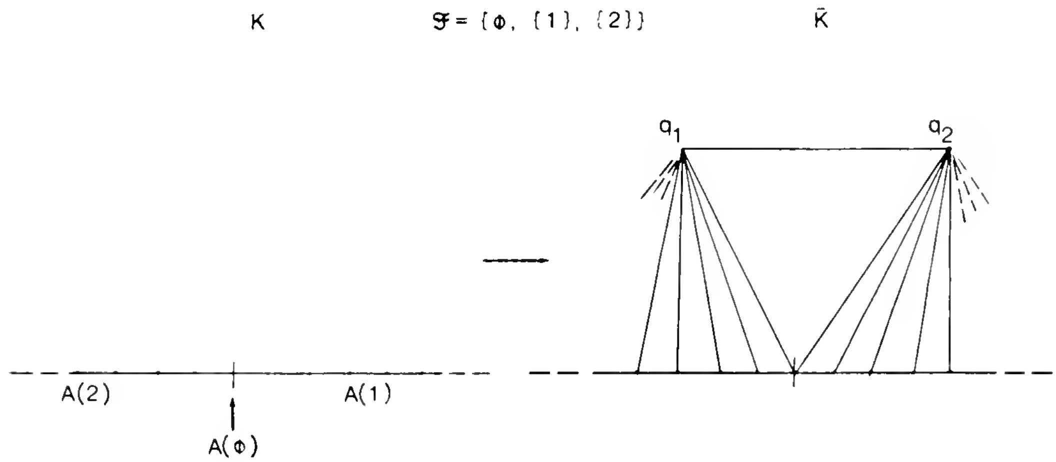


Figure 4(b)

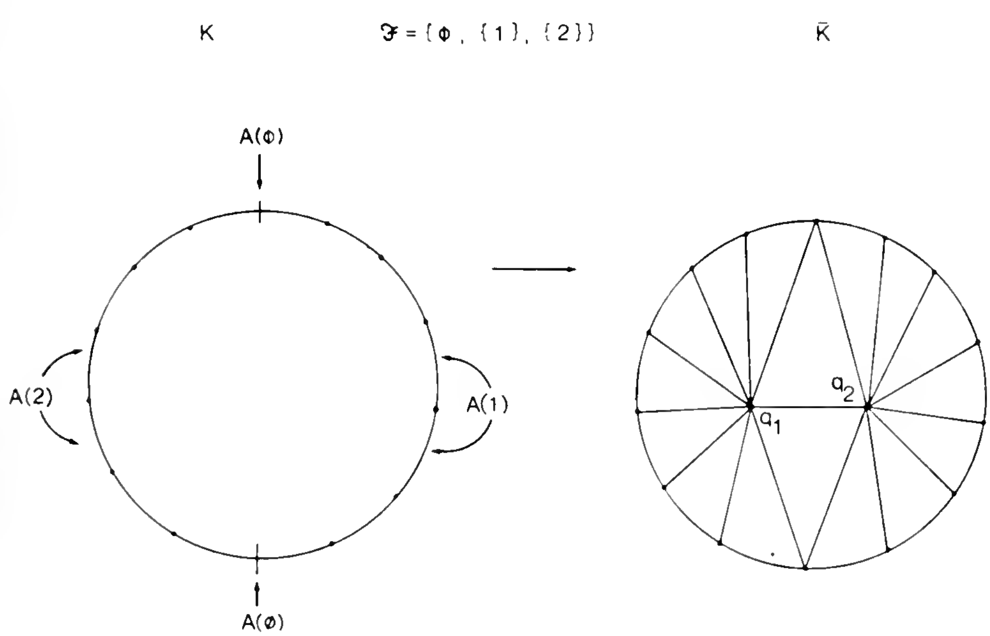


Figure 4 (c)

PROOF. We first prove that $S_1 \subset \partial\bar{K}$. Let $\bar{y} = y \cup Q_y$ be a maximal element of $S_1$. Then $y \in \partial' A(T_y)$ and $Q_y = Q_{T_y}$. The only n-simplex that contains $\bar{y}$ is of the form $y \cup \{v\} \cup Q_y$, where v is uniquely determined since $y \in \partial' A(T_y)$. Therefore $S_1 \subset \partial\bar{K}$.

Next we prove that $S_2 \subset \partial\bar{K}$. Let $\bar{y} = y \cup Q_y$ be a maximal element of $S_2$. Then y is full and $Q_y = Q_{T_y} \setminus \{q_i\}$ for some i, where $T_y \cup \{i\} \notin \mathfrak{J}$. Let $\bar{x} = \bar{y} \cup \{\alpha\}$ be an n-simplex in $\bar{K}$. We need to show $\alpha$ is uniquely determined. We cannot have $\alpha \in K^0$, since the set $T_y \cup \{i\} \notin \mathfrak{J}$. Hence $\alpha = q_j$ for some j. Suppose $j \neq i$. Then $T_y \cup \{i\} \setminus \{j\} \in \mathfrak{J}$, and in fact $T_y = T_y \cup \{i\} \setminus \{j\}$, whereby $j = i$, a contradiction. Therefore $\alpha = q_i$, and so $S_2 \subset \partial\bar{K}$. Therefore $S_1 \cup S_2 \subset \partial\bar{K}$.

Now let us prove the converse. Let $\bar{x} = x \cup Q_{T_x}$ be an n-simplex in $\bar{K}$ and let $\bar{y} \subset \bar{x}$ be an element of $\partial\bar{K}$, where $\bar{y} = y \cup Q$. We have two cases:

Case 1. $x = y \cup \{v\}$ for some $v \notin y$. Clearly we must have $y \in \partial A(T_x)$. If y were full, then $\bar{y} = y \cup Q_{T_y}$, and so $y \notin \partial\bar{K}$. Therefore y is not full. Hence $\bar{y} \in S_1$.

Case 2. $Q_{T_x} = Q \cup \{q_j\}$ for some $q_j \notin Q$. Suppose $N \setminus \{i | q_i \in Q\} \in \mathfrak{J}$. This means that $T_x \cup \{j\} \in \mathfrak{J}$. But then, by property (viii) of V-complexes, there is a unique $v \in K^0$ s.t. $y \cup \{v\} \in A(T_x \cup \{j\})$. Hence $\bar{y} = y \cup \{v\} \cup Q \in \bar{K}$, and hence $\bar{y} \notin \partial\bar{K}$, a contradiction. Therefore $N \setminus \{i | q_i \in Q\} \notin \mathfrak{J}$, and $\bar{y} \in S_2$.

Therefore $\partial\bar{K} \subset S_1 \cup S_2$, so $\partial\bar{K} = S_1 \cup S_2$. ⊗

Labelling Vertices and Adjacency on H-Complexes

Let $K$ be a V-complex and $\bar{K}$ its associated H-complex. Let $L(.)\colon K^0 \to N$ be a labelling function on $K^0$. We extend $L(.)$ to $\bar{K}^0$ by the simple rule that $L(q_i) = i$ for each $q_i \in \bar{Q}$, thereby obtaining a labelling function on $\bar{K}^0$. Let $\bar{x}$ be a simplex in $\bar{K}$. We define $L(\bar{x}) = \bigcup_{v \in \bar{x}} L(v)$.

We define two distinct n-simplices $\bar{x}, \bar{y} \in \bar{K}$ to be adjacent (written $\bar{x} \sim \bar{y}$) if

i) $\bar{x}$ and $\bar{y}$ are n-simplices

and ii) $L(\bar{x} \cap \bar{y}) = N$

The above definition of adjacency is quite standard for labelling functions on pseudomanifolds (see Gould and Tolle [8] or Lemke and Grotzinger [18]). Note that if $\bar{x} \sim \bar{y}$, $\bar{x}$ and $\bar{y}$ must be neighbors.

Characterization of Paths on H-Complexes

Let $K$ be a V-complex, $\bar{K}$ its associated H-complex, and let $L(.)$ be a labelling function on $K$, extended to $\bar{K}$. The following theorem, whose proof we omit, follows from the standard "ghost story" argument of complementary pivot theory (see Lemke [17], Gould and Tolle [8], Kuhn [11], Eaves [1], or Scarf [21].

Theorem 4. Let $\bar{x}$ be an n-simplex of $\bar{K}$. Then $\bar{x}$ is adjacent to at most two other n-simplices of $\bar{K}$. If $\bar{x}$ is adjacent to only one n-simplex of $\bar{K}$, then there is a unique (n-1)-simplex $\bar{y} \subset \bar{x}$ such that $L(\bar{y}) = N$ and $\bar{y} \in \partial\bar{K}$. ⊗

We define $\bar{B} = \{\bar{x} \in S_1 | L(\bar{x}) = N\}$ and $\bar{G} = \{\bar{x} \in S_2 | L(\bar{x}) = N\}$.

The notations $\bar{B}$ and $\bar{G}$ are short for "bad" and "good". In most applications, a path-following scheme will terminate with an element of $\bar{B}$ or $\bar{G}$. $\bar{G}$ typically contains those simplices with pre-specified desirable properties, whereas $\bar{B}$ does not.

Proposition 5. $\bar{B} \cap \bar{G} = \emptyset$.

PROOF. Suppose $\bar{x} \in \bar{G}$. Then $|\bar{x}| = n$, so $\bar{x}$ is a maximal element of $S_2$. We can write $\bar{x} = x \cup Q$, where $x$ is full. But then $\bar{x} \notin S_1$, since otherwise $x$ is not full. Therefore $\bar{x} \notin \bar{B}$. ⊗

With the help of Theorem 4, we can construct and characterize "paths" on $\bar{K}$. Let $\{\bar{x}_i\}_i$ be a maximal sequence of n-simplices of $\bar{K}$ such that $L(\bar{x}_i) = N$, $\bar{x}_i \sim \bar{x}_{i+1}$ and $\bar{x}_{i-1} \neq \bar{x}_{i+1}$ for any i. If $\bar{x}_k$ is a right-endpoint of the sequence, define $\bar{x}_{k+1}$ to be the unique subset of $\bar{x}_k$ such that $L(\bar{x}_{k+1}) = N$ and $\bar{x}_{k+1} \in \partial\bar{K}$. If $\bar{x}_k$ is a left-endpoint of the sequence, define $\bar{x}_{k-1}$ to be the unique subset of $\bar{x}_k$ such that $L(\bar{x}_{k-1}) = N$ and $\bar{x}_{k-1} \in \partial\bar{K}$. The new sequence, with possible endpoints added, is called a path on $\bar{K}$. Note that endpoints are elements of $\bar{G} \cup \bar{B}$.

We can characterize paths on $\bar{K}$ as one of six types.

Type I. $\{\bar{x}_i\}_i$ where the sequence has no endpoints, and

i) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

ii) $\bar{x}_i \neq \bar{x}_j$ for any $i \neq j$.

Type II. $\{\bar{x}_i\}_i$ where the sequence has no endpoints, and

i) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

ii) $\bar{x}_{i-1} \neq \bar{x}_{i+1}$ for all $-\infty < i < +\infty$

iii) There is an $m>2$ such that $\bar{x}_i = \bar{x}_{i+m}$ for all $-\infty < i < +\infty$

iv) $\bar{x}_i \neq \bar{x}_{i+k}$ for any $0<k<m$.

Type III. $\{\bar{x}_i\}_i$ where the sequence consists of only three elements, say $\bar{x}_0$, $\bar{x}_1$, $\bar{x}_2$, and

i) $\bar{x}_0, \bar{x}_2 \in \bar{G} \cup \bar{B}$, $\bar{x}_0 \neq \bar{x}_2$

ii) $L(\bar{x}_1) = N$

iii) $\bar{x}_1$ is an n-simplex and $\bar{x}_0, \bar{x}_2 \subset \bar{x}_1$.

Type IV. $\{\bar{x}_i\}_i$ has more than three elements, and has two endpoints, say $\bar{x}_0$ and $\bar{x}_m$, and

i) $\bar{x}_0, \bar{x}_m \in \bar{G} \cup \bar{B}$, and $\bar{x}_0 \neq \bar{x}_m$

ii) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $0 < i < m-1$

iii) $\bar{x}_i \neq \bar{x}_j$ for any $i \neq j$, $0 \leq i,j \leq m$.

Type V. $\{\bar{x}_i\}_i$ has only a left endpoint, say $\bar{x}_0$, and

i) $\bar{x}_0 \in \bar{G} \cup \bar{B}$

ii) $\bar{x}_i \sim \bar{x}_{i+1}$ for all $i > 0$

iii) $x_i \neq x_j$ for all $i, j \geq 0$, $i \neq j$

Type VI. $\{\bar{x}_i\}_i$ has only a right endpoint, say $\bar{x}_m$, and

i) $\bar{x}_m \in \bar{G} \cup \bar{B}$

ii) $\bar{x}_{i-1} \sim \bar{x}_i$ for all $i < m$

iii) $\bar{x}_i \neq \bar{x}_j$ for all $i, j \leq m$, $i \neq j$.

A type I path stretches infinitely in both directions. A type II path is a loop. A type III path is a "degenerate" path consisting of one n-simplex and two of its (n-1) subsimplices. A type IV path is a path with two endpoints. A Type V or Type VI path consists of one endpoint and stretches infinitely in one direction.

In the applications of V-complexes and H-complexes, it is the endpoints of paths that are of interest. We have the following lemma .

Lemma 6. Let $\bar{x} \in \bar{K}$. Then $\bar{x}$ is an endpoint of a path if and only if $\bar{x} \in \bar{G} \cup \bar{B}$.

PROOF. If $\bar{x}$ is an endpoint of a path, by definition $\bar{x} \in \bar{G} \cup \bar{B}$. Conversely, let $\bar{x} \in \bar{G} \cup \bar{B}$. There is a unique n-simplex $\bar{z} = \bar{x} \cup \{\alpha\}$ for some $\alpha \in \bar{K}^0$, and $L(\bar{z}) = N$. We can construct a path starting at $\bar{x} = \bar{x}_0$, $\bar{x}_1 = \bar{z}$, etc. ⊗

Corollary 7. If $\bar{K}$ is finite, $\bar{B}$ and $\bar{G}$ have the same parity.

PROOF. If $\bar{K}$ is finite, the total number of endpoints of paths is finite and even. Each endpoint is in exactly one of the two sets above; hence, they have the same parity. ⊗

## Characterization of Paths on V-Complexes

The characterization of paths on V-complexes is achieved by establishing certain equivalence relationships between V-complexes and H-complexes. The first equivalence is given in the following lemma.

Lemma 8. Let $\bar{x}$ and $\bar{y}$ be n-simplices of $\bar{K}$. Let $\bar{x} = x \cup Q_x$, $\bar{y} = y \cup Q_y$. Then $\bar{x} \sim \bar{y}$ if and only if $x \sim y$.

PROOF. Suppose $\bar{x} \sim \bar{y}$. This means $L(\bar{x} \cap \bar{y}) = N$. We have

$$N = L(\bar{x} \cap \bar{y}) = L(x \cap y) \cup L(Q_x \cap Q_y) = L(x \cap y) \cup ((N \setminus (T_x) \cup (N \setminus T_y)).$$

Therefore

$$L(x \cap y) = N \setminus ((N \setminus T_x) \cap (N \setminus T_y)) = N \setminus (N \setminus (T_x \cup T_y)) = T_x \cup T_y.$$

Thus we see that $L(x \cap y) = T_x \cup T_y$, and so $x \sim y$. The same argument in reverse shows that if $x \sim y$, then $\bar{x} \sim \bar{y}$. ⊗

Define $G = \{x \in K | x$ is full, $L(x) \supset T_x$, and $L(x) \notin \mathcal{J}\}$.

$G$ can be thought of as the goal set, for in most applications of V-complexes, the algorithm searches for an element of $G$.

Define $B = \{x \in K \mid x \in \partial' A(T_x)$, and $L(x) = T_x\}$. We have the following lemmas:

Lemma 9. Let $x \in K$, and let $\bar{x} = x \cup Q_{T_x}$. Then $x \in B$ if and only if $\bar{x} \in \bar{B}$.

PROOF. Let $x \in B$. $L(x) = T_x$. $L(\bar{x}) = L(x) \cup L(Q_{T_x}) = T_x \cup (N \setminus T_x) = N$. Furthermore, $x \in \partial' A(T_x)$, so $\bar{x} \in S_1$. Therefore $\bar{x} \in \bar{B}$.

Conversely let $\bar{x} \in \bar{B}$. Then $x \in \partial' A(T_x)$ and $L(x) = N \setminus L(Q_{T_x}) = N \setminus (N \setminus T_x) = T_x$, whence $x \in B$. ⊗

Lemma 10. Let $x \in K$ and let $\bar{x} = x \cup Q_{L(x)}$. Then $x \in G$ if and only if $\bar{x} \in \bar{G}$.

PROOF. Let $x \in G$. Then $L(x) = T_x \cup \{j\}$ for some $j \notin T_x$, where $T_x \cup \{j\} \notin \mathfrak{J}$. $Q_{L(x)} = Q_{T_x} \setminus \{q_j\} \subset Q_{T_x}$, so $\bar{x} \in \bar{K}$. Also, $N \setminus \{i \mid q_i \in Q_{L(x)}\} = N \setminus (N \setminus (T_x \cup \{j\})) = T_x \cup \{j\} \notin \mathfrak{J}$. Furthermore, $L(\bar{x}) = L(x) \cup L(Q_{L(x)}) = L(x) \cup (N \setminus (L(x)) = N$. Therefore $\bar{x} \in \bar{G}$.

Conversely, let $\bar{x} \in \bar{G}$. Then $L(x) = N \setminus \{i \mid q_i \in Q_{L(x)}\} \notin \mathfrak{J}$. Hence $x \in G$. ⊗

Let $x \in K$ be full. We define the degree of $x$, written $\deg(x)$, to be the number of distinct simplices of $K$ adjacent to $x$.

Lemma 11. For any $x \in K$, $\deg(x) \leq 2$.

PROOF. From Lemma 8, we have $x \sim y$ if and only if $\bar{x} \sim \bar{y}$, where $\bar{x} = x \cup Q_{T_x}$ and $\bar{y} = y \cup Q_{T_y}$. Since $\bar{x}$ is adjacent to at most two simplices, so is $x$. ⊗

With the help of Lemma 11, we can construct paths on $K$. Let $\langle x_i \rangle_i$ be a maximal sequence of full simplices in $K$ such that $x_i \sim x_{i+1}$, $x_{i-1} \neq x_{i+1}$.

Let $x_k$ be a left endpoint of the sequence. Then $\bar{x}_k$ is a left endpoint of the associated sequence in $\bar{K}$. Define $\bar{x}_{k-1}$ as in the last section and define $x_{k-1}$ such that $\bar{x}_{k-1} = x_{k-1} \cup Q$ for appropriate $Q \subset \bar{Q}$. Likewise, if $\bar{x}_k$ is a right endpoint, define $x_{k+1}$ analogously. The new sequence, with possible endpoints added, is a <u>path</u> on $K$.

<u>Lemma 12</u>. Let $x \in K$. Then $x$ is an endpoint of a path on $K$ if and only if $x \in G \cup B$.

<u>PROOF</u>. Let $x$ be an endpoint of a path on $K$. Then $\bar{x} = x \cup Q$ (for appropriate choice of $Q \subset \bar{Q}$) is an endpoint of a path on $\bar{K}$. $\bar{x} \in \bar{G} \cup \bar{B}$. So $x \in G \cup B$. ⊗

<u>Lemma 13</u>. If $K$ if finite, $B$ and $G$ have the same parity.

<u>PROOF</u>. $B$ and $G$, by definition, have no simplices in common. There is a one-to-one correspondence between elements of $B$ ($G$) and elements of $\bar{B}$ ($\bar{G}$). Also, if $K$ is finite, so is $\bar{K}$. Thus, by Corollary 7, $B$ and $G$ have the same parity. ⊗

We thus see a complete equivalence between paths on $K$ and on $\bar{K}$. Hence we can classify paths on $K$ as one of six types.

<u>Type I</u>. $\langle x_i \rangle_i$, where the sequence has no endpoints, and

i) $x_i \sim x_{i+1}$ for all $i$

ii) $x_i \neq x_j$ for all $i \neq j$.

<u>Type II</u>. $\langle x_i \rangle_i$, where the sequence has no endpoints, and

i) $x_i \sim x_{i+1}$ for all $i$

ii) $x_{i-1} \neq x_{i+1}$ for all $i$

iii) There is an $m > 2$ such that $x_i = x_{i+m}$ for all $i$

iv) $x_i \neq x_{i+k}$ for all $i$, all $0 < k < m$.

<u>Type III</u>. $\langle x_i \rangle_i$, where the sequence consists of only three elements, say $x_0$, $x_1$, $x_2$, and

i) $x_0, x_2 \in G \cup B$

ii) $L(x_1) \supset T_x$

iii) $x_1$ is full and $x_0, x_2 \subset x_1$.

<u>Type IV</u>. $\langle x_i \rangle_i$ has more than three elements, and has two endpoints, say $x_0$ and $x_m$, and

i) $x_0, x_m \in G \cup B$ and $x_0 \neq x_m$

ii) $x_i \sim x_{i+1}$ for all $0 < i < m-1$

iii) $x_i \neq x_j$ for any $i \neq j$, $0 \leq i, j \leq m$.

<u>Type V.</u> $\langle x_i \rangle_i$ has only a left endpoint, say $x_0$, and

i) $x_0 \in G \cup B$

ii) $x_i \sim x_{i+1}$ for all $i > 0$

iii) $x_i \neq x_j$ for any $i, j \geq 0$, $i \neq j$

<u>Type VI</u>. $\langle x_i \rangle_i$ has only a right endpoint, say $x_m$, and

i) $x_m \in G \cup B$

ii) $x_{i-1} \sim x_i$ for all $i < m$

iii) $x_i \neq x_j$ for all $i, j \leq m$, $i \neq j$

There are two ways to develop an algorithm based on a V-complex and a labelling function $L(.)$, depending on the nature of the set $A(\emptyset)$.

If $A(\emptyset)$ consists of a single 0-simplex, say $\{w\}$ and the empty set $\emptyset$, then $\emptyset \in B$, since $\emptyset \in \partial' A(\emptyset)$ and $L(\emptyset) = \emptyset = T_{\emptyset}$. Thus our algorithm consists of following a path whose endpoint is $\emptyset$.

If $A(\emptyset)$ consists of two 0-simplices, say $\{v\}$ and $\{w\}$, we have $v \sim w$, since $L(\{v\} \cap \{w\}) = L(\emptyset) = \emptyset = \emptyset \cup \emptyset = T_{\{v\}} \cup T_{\{w\}}$. Thus the algorithm consists of following the path containing $\{v\}$ and $\{w\}$ in one or both directions.

The purpose of the preceding exposition has been to show how to construct and follow paths on a V-complex. We used the construction of an H-complex to expedite the development of the theory. It should be noted that the characterization of paths on a V-complex can be demonstrated without resorting to the H-complex superstructure. However, the introduction of the H-complex renders the proofs less cumbersome, and shows the equivalence of path-following on a V-complex and path-following on n-dimensional pseudomanifolds. The latter is an "ordinary" phenomenon familiar to most researchers in complementary pivot theory. When viewed properly, path following on a V-complex is equivalent to path-following on n-dimensional pseudomanifolds, and can be viewed as the "projection" onto $K$ of path-following on $\bar{K}$.

In most algorithms based on V-complexes, we search for an element of $G$. We have seen that the set $G$ is derived from the structural properties of $\mathcal{J}$, and hence the way our complex $K$ is divided up into the $A(T)$ is intimately connected to what we can expect to look for in an algorithm on $K$. Conversely, suppose we wish to find elements $x$ of $K$ with certain labels $L(x) \in G$, where $G$ is some set. If we can divide the space into $A(T)$, $T \in \mathcal{J}$, such that $G$ arises from $\mathcal{J}$, we are close to our stated purpose.

## Orientation

This section develops an orientation theory on V- and H-complexes. At issue are conditions which guarantee that paths of simplices have certain orientation properties. This is accomplished in theorems 27 and 28, which give necessary and sufficient conditions on a V-complex for its associated H-complex to be orientable. The orientation theory is developed in the context of H-complexes, which are pseudomanifolds. This material could be developed for V-complexes without explicit reference to the associated H-complex. However, the development would be much more cumbersome, and would not show the implicit equivalence with orientation on pseudomanifolds.

As this material is not central to this study, and the subsequent development is not very elegant, the reader can omit this section without detracting from the exposition.

The use of orientation in complementary pivot schemes was first developed by Shapley [24] for the linear complementarity problem, advanced by Eaves and Scarf [3] and Eaves [2] for subdivided manifolds, and extended to pseudomanifolds by Lemke and Grotzinger [18]. An extensive treatment of orientation in the context of complementary pivot algorithms is presented in [28]. Our set-up is slightly different than that of Lemke and Grotzinger; however, the interested reader can easily establish the similarity.

Pivots and C-Pivots on Pseudomanifolds

Let $\bar{K}$ be an orientable H-complex of dimension n, oriented by Or(.), with vertex set $\bar{K}^0$. Let $N = \{1, \ldots, n\}$ and let $L(\cdot):\bar{K}^0 \to N$. We define the set

$$D = \{\bar{x} \in \bar{K} \,|\, |\bar{x}| = n+1,\ L(\bar{x}) = N\} \bigcup \{\bar{x} \in \partial\bar{K} \,|\, |\bar{x}| = n,\ L(\bar{x}) = N\}.$$

D consists then of n-simplices of $\bar{K}$ whose labels exhaust N, and simplicies on the boundary of $\bar{K}$ whose labels exhaust N. We remark that the two sets above whose union is D are disjoint. Denote these sets by $D_1$ and $D_2$, respectively.

Let $\bar{x} \in D$. There is a very natural way to order the elements of $\bar{x}$. If $\bar{x} \in D_1$, we can write $\bar{x} = \{v_0, \ldots, v_n\}$. The ordering $(v_{i_0}, \ldots, v_{i_n})$ of x is called a C-ordering if and only if:

$$L(v_{i_j}) = j, \qquad j = 1, \ldots, n.$$

Note there are always two C-orderings of $\bar{x}$. The reason for this is that among the labels of $\bar{x}$, there is some unique $r \in N$ such that two vertices of $\bar{x}$ have the label r. For $j \in N\setminus\{r\}$, the $j^{\underline{th}}$ component of a C-ordering of $\bar{x}$ must be the unique vertex $v_{i_j} \in \bar{x}$ for which $L(v_{i_j}) = j$. Denote by v' and v" those two vertices in $\bar{x}$ whose labels are r. Then the two C-orderings of $\bar{x}$ are:

$$(v'', v_{i_1}, \ldots, v_{i_{r-1}}, v', v_{i_{r+1}}, \ldots, v_{i_n})$$

and

$$(v', v_{i_1}, \ldots, v_{i_{r-1}}, v'', v_{i_{r+1}}, \ldots, v_{i_n})$$

Also note that these two orderings have opposite orientations, i.e. one is (+) and the other is (-).

If $\bar{x} \in D_2$, we can write $\bar{x} = \{v_i, \ldots, v_n\}$. The ordering $(v_{i_1}, \ldots, v_{i_n})$ is called a C-ordering if and only if

$$L(v_{i_j}) = j, \qquad j = 1, \ldots, n.$$

The C-ordering for $\bar{x} \in D_2$ is unique.

With the notion of a pivot in mind, we now define a C-pivot on elements of D. For $x \in D_1$, let $(v_{i_0}, \ldots, v_{i_n})$ be a C-ordering of x. A C-pivot is performed on $\bar{x}$ as follows:

Case 1. $\{v_{i_1}, \ldots, v_{i_n}\} \in \partial\bar{K}$. In this case, simply drop $v_{i_0}$ from $\bar{x}$, and let $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}\}$. The derived ordering on $\bar{y}$ is $(v_{i_1}, \ldots, v_{i_n})$.

Case 2. $\{v_{i_1}, \ldots, v_{i_n}\} \notin \partial\bar{K}$. In this case, there is a unique $\bar{v} \in \bar{K}^0$, $\bar{v} \neq v_{i_0}$ such that $\{v_{i_1}, \ldots, v_{i_n}, \bar{v}\} \in \bar{K}$. $L(\bar{v}) = r$ for some $r \in N$. Set $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}, \bar{v}\}$ and form the new ordering $(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$ of $\bar{y}$.

If $\bar{x} \in D_2$, let $(v_{i_1}, \ldots, v_{i_n})$ be the C-ordering of $\bar{x}$. A C-pivot on $\bar{x}$ is performed as follows :

Let $\bar{v}$ be the unique element of $\bar{K}^0$ such that $\bar{x} \cup \{\bar{v}\}$ is an n-simplex of $\bar{K}$. $L(\bar{v}) = r$ for some $r \in N$. Set $\bar{y} = \{v_{i_1}, \ldots, v_{i_n}, \bar{v}\}$ and from the new ordering $(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$ of $\bar{y}$.

We have the following results on C-pivots:

Proposition 14. Let $\bar{y}$ be derived from a C-pivot on $\bar{x} \in D_1$. Then the ordering of $\bar{y}$ is a C-ordering and the orderings on $\bar{x}$ and $\bar{y}$ as specified in the C-pivot have the same orientation.

PROOF. The first conclusion of the proposition follows immediately from the ordering defined on $\bar{y}$. The second conclusion follows from a case analysis.

Case 1. $\bar{y} \in \partial\bar{K}$. Then $Or(v_{i_0}, \ldots, v_{i_n}) = (-1)^0\, Or(v_{i_1}, \ldots, v_{i_n}) = Or(v_{i_1}, \ldots, v_{i_n})$

Case 2. $\bar{y} \notin \partial\bar{K}$. Then

$$Or(v_{i_0}, \ldots, v_{i_n}) = -Or(\bar{v}, v_{i_1}, \ldots, v_{i_n})$$

$$= Or(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n}). \otimes$$

Proposition 15. Let $\bar{y}$ be derived from a C∂pivot on $\bar{x} \in D_2$. Then the ordering of $\bar{y}$ is a C-ordering and the orderings on $\bar{x}$ and $\bar{y}$, as specified in the C-pivot, have opposite orientation.

PROOF. The first conclusion follows directly from the ordering fixed on $\bar{y}$. For the second conclusion, note that

$$Or(v_{i_r}, v_{i_1}, \ldots, v_{i_{r-1}}, \bar{v}, v_{i_{r+1}}, \ldots, v_{i_n})$$

$$= -Or(\bar{v}, v_{i_1}, \ldots, v_{i_n}) = -Or(v_{i_1}, \ldots, v_{i_n}). \otimes$$

Orientations on Paths Generated by C-Pivots

Let $\bar{K}$ be an orientable H-complex oriented by $Or(\cdot)$, $\bar{K}^0$ its vertex set, and assume, without loss of generality, that $N = \{1, \ldots, n\}$. Let $L(\cdot):\bar{K}^0 \to N$ be a labelling function. Let $\langle \bar{x}_i \rangle$ be a path on $\bar{K}$, possibly without left and/or right endpoints.

Choose $\bar{x}$ an element of the path. Note that $L(\bar{x}) = N$. If $\bar{x}$ is an endpoint of the path (say a left endpoint, and we can assume $\bar{x} = \bar{x}_0$, without loss of generality), there is a unique C-ordering of $\bar{x}_0$. Let $\bar{y}$ be derived from $\bar{x}_0$ by a C-pivot on $\bar{x}_0$. Then $\bar{y} = \bar{x}_1$, and $Or(\bar{x}_1) = -Or(\bar{x}_0)$ from Proposition 15. We can keep performing C-pivots on the $\bar{x}_i$, until we reach the right endpoint of the path, if it exists. For each of these pivots, we have $Or(\bar{x}_{i+1}) = Or(\bar{x}_i)$ by Proposition 14. We have just proved the following

<u>Lemma 16</u>. Let $\langle \bar{x}_i \rangle_i$ be a path with left endpoint $\bar{x}_0$. If $\bar{x}_{i+1}$ is obtained from $\bar{x}_i$ by a C-pivot, $Or(\bar{x}_0) = -Or(\bar{x}_i)$ for all $i > 0$. ⊗

In particular, we have

<u>Corollary 17</u>. Let $\langle \bar{x}_i \rangle_i$ be a path with left- and right-endpoints $\bar{x}_0$ and $\bar{x}_m$, generated by a series of C-pivots starting at $\bar{x}_0$. Then

i) $Or(\bar{x}_0) = -Or(\bar{x}_m)$,

and

ii) $Or(\bar{x}_i) = Or(\bar{x}_j)$ for all $0 < i, \; j \le m$. ⊗

Corollary 17 is analogous to other path orientation theorems presented elsewhere, see, for example, Shapley [23], Eaves and Scarf [3], Eaves [2], and Lemke and Grotzinger [18]. All of these theorems assert that the orientation along a path is constant except at the endpoints, whose orientations are opposite in sign.

Now suppose that $\bar{x}_i$ is an element of the path $\langle \bar{x}_i \rangle_i$ and $\bar{x}_i$ is not an endpoint. Then $\bar{x}_i \in D_1$. Since $L(\bar{x}) = N$, we can choose two C-orderings of $\bar{x}$, each one opposite in sign. C-pivoting on one of these orderings will yield $\bar{x}_{i+1}$, and the C-ordering of $\bar{x}_{i+1}$ will have the same

orientation as the C-ordering of $\bar{x}_i$. Continuing the C-pivot process, we will generate the path elements $\bar{x}_i$, $\bar{x}_{i+1}$, $\bar{x}_{i+2}$, ..., terminating if and only if $\langle \bar{x}_i \rangle_i$ has a right endpoint. By Proposition 14, $Or(\bar{x}_j) = Or(\bar{x}_i)$ for all $j > i$. A parallel argument for the other C-ordering completes the proof of the following.

Lemma 18. Let $\langle \bar{x}_i \rangle_i$ be a path on $\bar{K}$ and let $\bar{x}_i$ be an element of this path that is not an endpoint. Let the entire path be generated from $\bar{x}_i$ by its two C-orderings. We have $Or(\bar{x}_j) = -Or(\bar{x}_k)$ for all $j < i < k$. ⊗

In particular, we have

Corollary 19. Let $\langle \bar{x}_i \rangle_i$ be a path on $\bar{K}$ with left and right endpoints $\bar{x}_0$ and $\bar{x}_m$, respectively. If this path is generated from $\bar{x}_i$, $0 < i < m$, by the two C-orderings of $\bar{x}_i$, then

i) $Or(\bar{x}_0) = -Or(\bar{x}_m)$,

and

ii) $Or(\bar{x}_j) = -Or(\bar{x}_k)$ for all $j < i < k$. ⊗

By way of concluding thus far, we remark that the usual path orientation results for manifolds carry over to H-complexes. Actually, they do more than this--they carry over to orientable n-pseudomanifolds. For the only properties of H-complexes used in these two sections was that $\bar{K}$ is an orientable n-pseudomanifold and that the label set $N$ contains $n$ elements.

Conditions for which an H-complex is Orientable

In this section we give conditions on $\bar{K}$ that guarantee that $\bar{K}$ is orientable. Let $K, \mathcal{J}, N$, and $A(\cdot)$ define a V-complex, and let $\bar{K}$ be the H-complex associated with $K$. Let $|N| = n$. Assume that

i) for each $T \in \mathcal{J}$, $A(T)$ is orientable, and hence homogeneous, and

ii) for all $S, T \in \mathcal{J}$, there is a sequence $S_0, \ldots, S_m$ such that $S_i \in \mathcal{J}$, $i = 0, \ldots, m$, $S_0 = S$, $S_m = T$, and $|S_{i-1} \Delta S_i| \leq 1$, $i = 1, \ldots, m$.

We will show that $K, \mathcal{J}, N$, and $A(\cdot)$ satisfy the above two assumptions if and only if $\bar{K}$ is orientable.

Towards proving our main result, we make the following:

Definition. For $T \in \mathcal{J}$, define

$$\bar{A}(T) = \{x \cup Q \,|\, x \in A(T),\ Q \subset Q_T,\ x \cup Q \neq \emptyset\}.$$

$\bar{A}(T)$ can be thought of as a conical construction of $A(T)$ with each $q_i$, $i \notin T$. We have:

Lemma 20. $\bar{A}(T)$ is an orientable n-pseudomanifold.

PROOF. Clearly $\bar{A}(T)$ is closed under nonempty subsets. Let $x \cup Q \in \bar{A}(T)$. Then there is a $y$ in $A(T)$ such that $T_y = T$, since $A(T)$ is a pseudomanifold. Then $x \cup Q \subset y \cup Q_T$, and $|y \cup Q_T| = |T| + 1 + n - |T| = n+1$.

Any n-simplex of $\bar{A}(T)$ is of the form $y \cup Q_T$, where $y$ is a $|T|$-simplex in $A(T)$. Let $x \cup Q$ be an (n-1)-simplex in $A(T)$, that is a subset of $y \cup Q_T$. Suppose $x \cup Q \subset z \cup Q_z$, $|z \cup Q_z| = n+1$, and $z \cup Q_z \neq y \cup Q_T$. But then $Q_z = Q_T$, and since $A(T)$ is a pseudomanifold, the choice of $z$ is unique. This proves $\bar{A}(T)$ is an n-pseudomanifold.

Now let $x \cup Q_T$ and $y \cup Q_T$ be n-simplices in $\bar{A}(T)$. Then $x$ and $y$ are $|T|$-simplices in $A(T)$. Since $A(T)$ is homogeneous, there is a sequence $x = s_1, s_2, \ldots, s_m = y$ of $|T|$-simplices in $A(T)$ such that $|s_i \cap s_{i+1}| = |T|$, $i = 1, \ldots, m-1$. Then $x \cup Q_T = s_1 \cup Q_T$, $s_2 \cup Q_T$, $\ldots$, $s_m \cup Q_T = y \cup Q_T$ is a sequence of n-simplices in $\bar{A}(T)$ and $|(s_i \cup Q_T) \cap (s_{i+1} \cup Q_T)| = n$, $i = 1, \ldots, m-1$. Therefore $\bar{A}(T)$ is homogeneous.

Finally, we show that $\bar{A}(T)$ is orientable. Let $Or(\cdot)$ be a coherent orientation of the $|T|$-simplices of $A(T)$. Let $x \cup Q_T$ be an n-simplex of $\bar{A}(T)$. Let $|T| = t$. Order the vertices of $x \cup Q_T$, as $(v_0, \ldots, v_n)$, and let $p$ be the number of transpositions needed to transpose those $v_i \in Q_T$ to the last n-t places of the ordering, while preserving the local ordering of those $v_i \in A(T)$ and the local ordering of those $v_i \in Q_T$. Then we define $Or(v_0, \ldots, v_n) = (-1)^p \, Or(x)$.
If $y \in Q_T$ is obtained from $x \cup Q_T$ by a pivot, $Or(y \cup Q_T) = (-1)^p \, Or(y) = (-1)^p \, (-Or(x)) = -Or(x \cup Q_T)$. Thus $\bar{A}(T)$ is orientable. ⊗

Lemma 21. $\bar{K} = \bigcup_{T \in \mathfrak{I}} \bar{A}(T)$.

PROOF: Let $\bar{x} \in K$. Then we can write $\bar{x} = x \cup Q$ where $x \in K$, and $Q \subset \bar{Q}$, and $Q \subset Q_{T_x}$. But then $\bar{x} \in \bar{A}(T_x)$. Conversely, let $x \cup Q \in \bar{A}(T)$. Then $Q \subset Q_{T_x}$ and so $x \cup Q \in \bar{K}$. ⊗

Lemma 22. Any n-simplex of $\bar{K}$ is an element of exactly one $\bar{A}(T)$.

PROOF: Let $x \cup Q$ be an n-simplex in $\bar{K}$. Then $Q = Q_{T_x}$ and $x$ is full. Thus $x \cup Q \in \bar{A}(T_x)$. Suppose $x \cup Q \in \bar{A}(S)$ for some $S \in \mathfrak{I}$. Then $x \in A(S)$ and hence $S \supset T_x$. Also $Q_{T_x} \subset Q_S$ which implies $S \subset T_x$. Thus $S = T_x$. ⊗

Thus we see that as $T$ ranges over all elements of $\mathfrak{I}$, the $\bar{A}(T)$ partition $\bar{K}$ into "disjoint" n-pseudomanifolds. We use disjoint loosely since this partitioning only takes place among the n-simplices of $\bar{K}$.

Next we have

Proposition 23. $\bar{K}$ is homogeneous.

PROOF: Let $\bar{x}$ and $\bar{y}$ be n-simplices in $\bar{K}$. We can write $\bar{x} = x \cup Q_{T_x}$, $\bar{y} = y \cup Q_{T_y}$ for appropriate $x, y \in K$. By Assumption (ii), there is a sequence $T_x = T_1, \ldots, T_m = T_y$ such that $T_i \in \mathfrak{I}$, $i = 1, \ldots, m$, and $|T_i \Delta T_{i+1}| \leqq 1$, $i = 1, \ldots, m-1$.

We shall now show how to construct a sequence of neighboring simplices in $\bar{K}$ that have $\bar{x}$ and $\bar{y}$ as endpoints, using an induction argument on m. If $m = 1$, then such a sequence of neighboring simplices exists because $\bar{x}, \bar{y} \in \bar{A}(T) \subset \bar{K}$, and $\bar{A}(T)$ is homogeneous. Suppose a sequence of neighboring simplices $\langle s_i \rangle_{i=0}^{\ell}$ exists whose endpoints are $\bar{x}$ and $\bar{z} \in T_{m-1}$. Then either $T_m = T_{m-1} \cup \{k\}$ for some $k \notin T_{m-1}$, or $T_m = T_{m-1} \setminus \{k\}$ for some $k \in T_{m-1}$.

In the former case, $\tilde{z} = \bar{z} \setminus \{q_k\} \cup \{w\}$ is an n-simplex in $\bar{K}$, that is in $\bar{A}(T_m)$, for some unique $w \in \bar{K}^0$. Since $\bar{A}(T_m)$ is homogeneous, there is a sequence of neighboring n-simplices $\langle t_i \rangle_{i=0}^{j}$, where $t_0 = \tilde{z}$, $t_j = \bar{y}$. Thus the sequence

$$\bar{x} = s_0, \ldots, s\ = \bar{z}, \tilde{z} = t_0, \ldots, t_j = \bar{y}$$

of neighboring simplices has $\bar{x}$ and $\bar{y}$ as its endpoints. An analogous argument establishes the result when $T_m = T_{m-1} \setminus \{k\}$. ⊗

The next results will also be used in the proof that $\bar{K}$ is orientable.

<u>Proposition 24</u>. There is a unique set $T^* \in \mathcal{J}$ such that $S \in \mathcal{J}$ implies $S \supset T^*$.

<u>PROOF</u>. Define $T^* = \bigcap_{S \in \mathcal{J}} S$. Then $T^* \in \mathcal{J}$ and any $S \in \mathcal{J}$ contains $T^*$. Clearly $T^*$ is uniquely determined. ⊗

<u>Proposition 25</u>. Let $S, T \in \mathcal{J}$, $S \neq T$, $|S| = |T|$. Then $\bar{A}(S) \cap \bar{A}(T)$ contains no (n-1)-simplices.

<u>PROOF</u>. Let $x \cup Q \in \bar{A}(S) \cap \bar{A}(T)$. Let $t = |S| = |T|$. We have $x \in A(S)$, $x \in A(T)$, so $x \in A(S \cap T)$. But $|S \cap T| \leq t-1$. Thus $|x| \leq t$. Also $Q \subset Q_S \cap Q_T = Q_{S \cup T} \leq n - (t+1)$. Thus $|x \cup Q| \leq t + n - t - 1 \leq n - 1$. Therefore $x \cup Q$ cannot be an (n-1)-simplex. ⊗

We are now ready to describe an inductive procedure for orienting $\bar{K}$. Let $T^* \in \mathcal{J}$ be the set described in Proposition 24. Let $d = |T^*|$. Let $m = \max_{T \in \mathcal{J}} |T| - d$. Then we partition $\mathcal{J}$ into m+1 classes, $\mathcal{J}_d, \ldots, \mathcal{J}_{d+m}$, where $\mathcal{J}_k = \{T \in \mathcal{J} \mid |T| = k\}$. Note that $\mathcal{J} = \bigcup_{k=0}^{m} \mathcal{J}_{d+m}$, and for all $k \neq j$, $\mathcal{J}_k \cap \mathcal{J}_j = \emptyset$. Our procedure for orientating $\bar{K}$ is as follows:

Step 0. Orient $\bar{A}(T^*)$. Let $Or(\cdot)$ denote the orientation on $\bar{A}(T^*)$.

Set $\bar{K}_0 = \bar{A}(T^*)$.

Step i. $(i = 1, \ldots, m)$: Let $\bar{K}_i = \bar{K}_{i-1} \cup (\bigcup_{T \in \mathcal{J}_i} \bar{A}(T))$. Extend the orientation $Or(\cdot)$ to $\bar{K}_i$ by using the induced orientation on $\bar{K}_{i-1}$ to orient $\bar{A}(T)$, $T \in \mathcal{J}_i$.

We now show that each step of this procedure is executable and the result is a coherent orientation of $\bar{K}$. Note $\bar{K}_m = \bar{K}$. Our proof is as follows:

Clearly Step 0 is executable, since $\bar{A}(T^*)$ is orientable. Suppose steps 0, ..., i-1 are executable and result in a coherent orientation of $\bar{K}_{i-1}$. The following lemma serves as a basis for our proof:

Lemma 26. Suppose $T' \in \mathcal{J}_i$. Then $\bar{K}_{i-1} \cap \bar{A}(T')$ is an orientable (n-1)-pseudomanifold and is a subset of $\partial\bar{K}_{i-1}$ and $\partial\bar{A}(T')$.

PROOF. By the induction hypothesis $\bar{K}_{i-1}$ is an orientable n-pseudomanifold. So is $\bar{A}(T')$. Let us denote $L = \bar{K}_{i-1} \cap \bar{A}(T')$ for notational convenience. $L$ then is closed under nonempty subsets, and so is a complex. Let $x \cup Q \in L$. Then $x \in A(T)$ for some $T$, $|T| < d+i$, and $x \in A(T')$, $Q \subset Q_{T'}$. By assumption (ii), there exists $k \in T'$ such that $x \in A(T' \setminus \{k\})$. Let $y \in A(T' \setminus \{k\})$ be full and contain $x$ $(x \subset y)$, such that $T_y = T' \setminus \{k\}$. Then $x \cup Q \subset y \cup Q_{T'}$. Note $y \cup Q_{T'} \in L$. Furthermore $|y \cup Q_{T'}| = d + i + n - (d + i) = n$. Thus every element of $L$ is a subset of an (n-1)-simplex of $L$.

Now let $x \cup Q$ be an (n-1)-simplex of $L$. From the preceding remarks, we know $T_x = T' \setminus \{k\}$ for some $k \in T'$, and $Q = Q_{T'}$. Let $x \cup Q \setminus \{\alpha\}$ be an (n-2)-simplex of $L$, and suppose $x \cup Q \setminus \{\alpha\} \cup \{\beta\}$ is an (n-1)-simplex of $L$, $\beta \neq \alpha$. We need to show that there is at most one choice of $\beta$. Clearly, $\alpha \notin Q$, so $\alpha \in x$. We have two cases:

Case I. $x \setminus \{\alpha\}$ is not full. Then since $x \setminus \{\alpha\} \cup \{\beta\}$ must be full, $\beta$ is the unique element of $K^0$ such that $x \setminus \{\alpha\} \cup \{\beta\}$ is a $(d+i-1)$-simplex of $A(T' \setminus \{k\})$.

Case 2. $x \setminus \{\alpha\}$ is full. In this case $T_{x \setminus \{\alpha\}} = T' \setminus \{k\} \setminus \{j\}$ for some $j \neq k$, $j \in T'$.
Since $x \setminus \{\alpha\} \cup \{\beta\}$ must be a full $(d+i-1)$-simplex, $\beta$ is the unique element of $K^0$ such that $x \setminus \{\alpha\} \cup \{\beta\}$ is a $(d+i-1)$-simplex of $A(T' \setminus \{j\})$.
Thus we see that $L$ is a pseudomanifold of dimension $(n-1)$.

Our next task is to show that $L$ is homogeneous. Let $x \cup Q_{T'}$, $y \cup Q_{T'}$ be distinct $(n-1)$-simplices in $L$. If $T_x = T_y$, then since $A(T_x)$ is homogeneous, there is a sequence $x = s_0, \ldots, s_k = y$ of neighbors such that each $s_j \in A(T_x)$, $j = 0, \ldots, k$. Then $x \cup Q_{T'} = s_0 \cup Q_{T'}, \ldots, s_j \cup Q_{T'}, \ldots, s_k \cup Q_{T'}$ is a sequence of neighbors in $L$. Suppose then $T_x \neq T_y$. $T_x = T' \setminus \{j\}$, $T_y = T' \setminus \{k\}$, for some $j, k$, where $j \neq k$, $j \in T'$, $k \in T'$. Let $z \in A(T' \setminus \{j\} \setminus \{k\})$. Then there exists $\alpha, \beta$ $K^0$ such that $z \cup \{\alpha\} \in A(T' \setminus \{j\})$, $z \cup \{\beta\} \in A(T' \setminus \{k\})$. Let $x = s_0, \ldots, s_p = z \cup \{\alpha\}$ be a sequence of neighbors in $A(T' \setminus \{j\})$, $z \cup \{\beta\} = t_0, \ldots, t_r = y$ a sequence of neighbors in $A(T' \setminus \{k\})$. Then the sequence $x \cup Q_{T'} = s_0 \cup Q_{T'}, \ldots, s_p \cup Q_{T'}$, $t_0 \cup Q_{T'}, \ldots, t_r \cup Q_{T'}$ is a sequence of neighbors in $L$. Thus $L$ is homogeneous.

Next we show that $L \subset \partial\bar{K}_{i-1}$. Let $x \cup Q_{T'}$ be an $(n-1)$-simplex of $L$. Since $T_x = T' \setminus \{k\}$ for some $k \in T'$, we can write $x \cup Q_{T'} = x \cup Q_{T_x} \setminus \{q_k\}$. Any $n$-simplex of $\bar{K}_{i-1}$ is of the form $y \cup Q_{T_y}$ where $|T| \leq d + i - 1$. Thus the unique $n$-simplex of $\bar{K}_{i-1}$ containing $x \cup Q_{T'}$ is $x \cup Q_{T_x}$, and hence $x \in \partial\bar{K}_{i-1}$. A similar argument shows that $L \supset \partial\bar{A}(T')$.

It only remains to show that $L$ is orientable. Since $\bar{K}_{i-1}$ is orientable, $Or(\cdot)$ on $\bar{K}_{i-1}$ induces an orientation $Or(\cdot)$ on $L \subset \partial\bar{K}_{i-1}$. We need to show that this induced orientation is coherent. Let $\bar{x}, \bar{y} \in L$ be neighbors. Let us assign labels to elements of $\bar{K}_{i-1}$ as follows: For $v \in \bar{K}_{i-1}$, $v \notin \bar{x} \cup \bar{y}$, let $L(v) = 1$. We can write $\bar{x} = \{v_1, \ldots, v_n\}$, $\bar{y} = \{v_{n+1}, v_2, \ldots, v_n\}$ and let $L(v_1) = L(v_{n+1}) = 1$, $L(v_i) = i$, $i = 2, \ldots, n$. Let us do C-pivots on the C-ordering of $\bar{x}$. This will trace a path of simplices of $\bar{K}_{i-1}$, which if it has a right endpoint, the right endpoint will be $\bar{y}$. Furthermore, by the nature of our labelling function, all elements of the path will contain $\bar{x} \cap \bar{y}$. At least one element of $\bar{x} \cap \bar{y}$ will be an element of $K^0$, and since $K$ is locally finite, the path will have a right endpoint. From the results of the first part of this section, $Or(\bar{x}) = -Or(\bar{y})$, thus establishing that $\cup Or(\cdot)$ is coherent on $L$. ⊗

With Lemma 26 established, we can orient $L$ using the induced orientation $Or(\cdot)$ from $\bar{K}_i$. Now let $\bar{x} = \{v_1, \ldots, v_n\}$ be a fixed ordered element of $L$. Since $\bar{x} \in \partial\bar{K}_{i-1}$, $\bar{x} \in \partial\bar{A}(T')$, there exist unique elements $\alpha, \beta \in \bar{K}^0$ such that $\{\alpha, v_1, \ldots, v_n\} \in \bar{K}_{i-1}$, $\{\beta, v_1, \ldots, v_n\} \in \bar{A}(T')$. Define $Or(\beta, v_1, \ldots, v_n) = -Or(\alpha, v_1, \ldots, v_n)$, and extend $Or(\cdot)$ to all of $\bar{A}(T')$ by using $\{\beta, v_1, \ldots, v_n\}$ as a "seed". This makes $\bar{A}(T')$ coherently oriented, and also $\bar{K}_{i-1} \cup \bar{A}(T')$ coherently oriented. We can repeat this procedure for all $T' \in \mathcal{J}_i$ since for any $S, T \in \mathcal{J}$ $\bar{A}(S) \cap \bar{A}(T)$ contains no (n-1)-simplices or n-simplices, i.e. $\bar{A}(S)$ and $\bar{A}(T)$ share no common boundary.

Thus step $i$, $i = 1, \ldots, m$, of our procedure is executable and results in a coherent orientation of $\bar{K}_i$. Hence $\bar{K}_m = \bar{K}$ is orientable. We have just proved:

Theorem 27. Let $A(\cdot), \mathfrak{J}$, N, K satisfy assumptions (i) and (ii) of this section. Then $\bar{K}$ is orientable. ⊗

We also have:

Theorem 28. Let $\bar{K}$ be orientable. Then $A(\cdot), \mathfrak{J}$, N, and K satisfy assumption (i) and (ii) of this section.

PROOF. Suppose assumption (i) does not hold. Then for some $T \in \mathfrak{J}$, $A(T)$ is not orientable. But then $\bar{A}(T)$, the conical construction of $A(T)$ with the $q_i$, $i \notin T$, is not orientable, whence $\bar{K} = \bigcup_{T \in \mathfrak{J}} \bar{A}(T)$ is not orientable, a contradiction. Thus assumption (i) is satisfied.

Suppose assumption (ii) does not hold. Then for some $S, T \in \mathfrak{J}$, there is no sequence $S = S_0, S_1, \ldots, S_m = T$, such that $|S_i \mathbin{\Delta} S_{i+1}| \leqq 1$ $i = 0, \ldots, m-1$, and $S_i \in \mathfrak{J}$, $i = 0, \ldots, m$. Let $\bar{x} \in \bar{A}(S)$, $\bar{y} \in \bar{A}(T)$. If $\bar{K}$ is orientable, it is homogeneous. Hence there is a sequence of n-simplices $\bar{x} = \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_\ell = \bar{y}$ such that $\bar{x}_i$ and $\bar{x}_{i+1}$ are neighbors. Let $S_i = T_x i$ where $\bar{x}_i = x_i \cup Q_{T_x i}$. Because $\bar{x}_i$ and $\bar{x}_{i+1}$ are neighbors, we must have $|S_i \mathbin{\Delta} S_{i+1}| \nleq 1$, a contradiction. Thus assumption (ii) is satisfied. ⊗

## Conclusion and Remarks

In Part I of this study, we have defined a V-complex and characterized paths induced on simplices of a V-complex by a labelling function $L(\cdot)$. We have also demonstrated necessary and sufficient conditions for the H-complex associated with a V-complex to be orientable. Part II of this study will use these results to present constructive algorithmic proofs of a variety of combinatorial lemmas in topology, some new, some old. These lemmas give rise to proofs of fixed point, antipodal point, and stationary point theorems, and to a new set covering theorem on the simplex.

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